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DOI: https://doi.org/10.1017/S0308210500003462

Originally published at:
Walker, Ch (2004). Asymptotic behaviour of liquid-liquid dispersions. Proceedings of the Royal Society of Edinburgh: Section A, 134(4):753-772.
DOI: https://doi.org/10.1017/S0308210500003462

Proceedings of the Royal Society of Edinburgh, **134A**, 753–772, *2004*

Asymptotic behaviour of liquid–liquid dispersions

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(MS received 23 September 2003; accepted 18 February 2004)

Based on earlier results on existence, we study the asymptotic behaviour of solutions to the coalescence-breakage equations, including the volume-scattering phenomenon and high-energy collisions. The solutions are shown to converge towards one particular equilibrium, provided the kernels satisfy a kind of reversibility. We also derive stability of these equilibria in a suitable topology.

1. Introduction

In the present article we consider the evolution of a liquid–liquid dispersion, which is a system formed by two immiscible liquids, where one of these liquids consists of a very large number of droplets that are finely distributed in the other one. These droplets undergo the influences of binary coalescence and binary breakage, meaning that two droplets can merge to build a larger droplet, or that a droplet can split into two smaller ones.

As opposed to most other models considered in literature, we take into account that droplets cannot become arbitrarily large and that experimental observations suggest the existence of a maximal droplet mass (or volume) beyond which no droplet can survive (see [22]). A particular model that paid attention to this feature was introduced for the first time by Fasano and Rosso [14] (see also [4, 13, 21]) and was then developed further by the author [27]. Such a maximal droplet size requires a new interaction mechanism, called *volume scattering*, to prevent the occurrence of droplets that are 'too large'. The underlying idea is that if two droplets collide that have a cumulative mass exceeding the maximal droplet mass, the virtual droplet is highly unstable and immediately decays into two droplets, both with mass within the admissible range.

Another new feature taken into consideration in our model is the possibility of high-energy collisions leading to a shattering of the involved droplets. Such a breakage mode has been contemplated in physical literature (cf. [7, 8, 29]), but—at least to the author's knowledge—only its discrete version has been investigated mathematically so far (see [20]).

We describe the evolution of the dispersion by means of the droplet-size distribution function $u = u(t, y)$ at time t (per unit mass), y being the mass (or volume) of a droplet. By $y_0 \in (0, \infty)$ we denote the maximal droplet mass, which we assume to be *a priori* known, so that $(0, y_0]$ represents the admissible range of droplet masses. Neglecting dependence on spatial coordinates (for a treatment of the spatially inhomogeneous case, we refer to [28]), the evolution of the system of droplets that undergo both coalescence and breakage can be described by the set of integro-differential equations

$$\left.\begin{aligned} \dot{u}(y) &= \varphi(u)L[u](y), \quad t>0, \quad y\in(0,y_0],\\ u(0,y) &= u^0(y), \qquad\qquad\quad y\in(0,y_0], \end{aligned}\right\} \tag{$*$}$$

where u^0 is a given initial distribution. The reaction terms are defined as

$$L[u] := L_{\mathrm{b}}[u] + L_{\mathrm{c}}[u] + L_{\mathrm{s}}[u],$$

whereby, for $y\in(0,y_0]$,

$$\begin{aligned} L_{\mathrm{b}}[u](y) &:= \int_y^{y_0} \gamma(y',y)u(y')\,\mathrm{d}y' - \tfrac{1}{2}u(y)\int_0^y \gamma(y,y')\,\mathrm{d}y',\\ L_{\mathrm{c}}[u](y) &:= \frac{1}{2}\int_0^y K(y',y-y')P(y',y-y')u(y')u(y-y')\,\mathrm{d}y'\\ &\quad + \frac{1}{2}\int_y^{y_0}\int_0^{y'} K(y'',y'-y'')Q(y'',y'-y'')\\ &\qquad\qquad \times \beta_{\mathrm{c}}(y',y)u(y'')u(y'-y'')\,\mathrm{d}y''\mathrm{d}y'\\ &\quad - u(y)\int_0^{y_0-y} K(y,y')\{P(y,y')+Q(y,y')\}u(y')\,\mathrm{d}y',\\ L_{\mathrm{s}}[u](y) &:= \frac{1}{2}\int_{y_0}^{y_0+y}\int_{y'-y_0}^{y_0} K(y'',y'-y'')\beta_{\mathrm{s}}(y',y)u(y'')u(y'-y'')\,\mathrm{d}y''\mathrm{d}y'\\ &\quad - u(y)\int_{y_0-y}^{y_0} K(y,y')u(y')\,\mathrm{d}y'. \end{aligned}$$

The linear operator $L_{\mathrm{b}}[u]$ gives the gain and loss of droplets of mass y due to binary breakage, where the kernel $\gamma(y,y')$ represents the rate at which a droplet of mass y decays into a droplet of mass $y'\in(0,y)$. Binary breakage in particular means that

$$\gamma(y,y') = \gamma(y,y-y'), \quad 0<y'<y\leqslant y_0. \tag{1.1}$$

When two droplets y and y' with cumulative mass $y+y'\leqslant y_0$ collide, three different events may arise, being described by the collision operator $L_{\mathrm{c}}[u]$. They either coalesce with probability $P(y,y')$, or a shattering of these droplets occurs with probability $Q(y,y')$, or just nothing happens, meaning that the droplets remain unchanged. Obviously, it then holds that

$$0\leqslant P(y,y')+Q(y,y')\leqslant 1, \quad 0<y+y'\leqslant y_0. \tag{1.2}$$

The symmetric function $K(y,y')$ denotes the rate of binary collision. Furthermore, $\beta_{\mathrm{c}}(y+y',y'')$ is the distribution function of products from a particle $y+y'\in(0,y_0]$ shattering after collision, and β_{c} satisfies

$$\beta_{\mathrm{c}}(y+y',y'') = \beta_{\mathrm{c}}(y+y',y+y'-y''), \quad 0<y''<y+y'\leqslant y_0. \tag{1.3}$$

The factors $\frac{1}{2}$ come in to compensate for double counting.

The scattering operator $L_{\mathrm{s}}[u]$ represents the interaction of two colliding droplets whose cumulative mass exceeds y_0 and who immediately split into two droplets, both with mass in $(0, y_0]$. The distribution function $\beta_{\mathrm{s}}(y+y', y'')$ for $y+y' \in (y_0, 2y_0]$ has an analogue meaning as $\beta_{\mathrm{c}}(y+y', y'')$ for $y+y' \in (0, y_0]$ above. Therefore,

$$\beta_{\mathrm{s}}(y+y', y'') = \beta_{\mathrm{s}}(y+y', y+y'-y''), \quad 0 < y+y'-y_0 \leqslant y'' \leqslant y_0. \tag{1.4}$$

We assume that β_{c} and β_{s} merely depend on the cumulative mass $y+y'$ of the colliding droplets, although there would barely be a difference in the further analysis to allow a dependence on each colliding droplet.

Finally, the efficiency factor $\varphi(u)$ linked to some average properties of the dispersion enhances or depresses the dynamics, while the mechanical structure of the interactions is described by the kernels γ, β_{c}, β_{s}, K, P, and Q. For instance, $\varphi(u)$ may be of the form

$$\varphi(u) = \Phi\biggl(\int_0^{y_0} u(y)\,\mathrm{d}y, \int_0^{y_0} y^{2/3}u(y)\,\mathrm{d}y\biggr), \tag{1.5}$$

where $\Phi : \mathbb{R}^2 \to \mathbb{R}^+$ is a given function. This means that $\varphi(u)$ is related to the total number of droplets and the total surface area. Clearly, no mathematically substantial differences arise if one considers for each process an individual efficiency factor. But to keep the notation simple, we omit this.

The model considered in [4,13,14,21] can be recovered from $(*)$ by putting $P \equiv 1$. In particular, the shattering terms then drop, since $Q \equiv 0$, according to (1.2). For these reduced equations, global existence and uniqueness of non-negative and mass-preserving solutions is shown in [14], which are Lipschitz continuous with respect to droplet size. These results are extended in [4] to include breakage kernels with singularities. Numerical simulations are performed in [21], exhibiting some interesting features concerning the qualitative behaviour of the solutions for large times.

Finally, a slightly modified version of model $(*)$ (also including multiple breakage) is considered by the author [27]. In the particular case of binary breakage, solutions belonging to the space $L_1((0, y_0])$ are shown to exist globally in time and to be unique.

It is the purpose of the present paper to investigate the long-time behaviour of the particular solutions of [27], assuming that the processes under consideration are somehow reversible. More precisely, we assume that the kernels satisfy an extended version of the so-called *detailed balance condition* (see hypothesis (H_6) below) guaranteeing the existence of equilibria and also providing a Lyapunov function. Such a reversibility condition on the kernels was used in various papers in order to study the qualitative behaviour of solutions for large times. For a treatment of the asymptotic behaviour of solutions to the discrete analogue of $(*)$, we refer to [5,6,10] concerning the spatially homogeneous case and to [9, 18] for the case including diffusion (see also [3,19] for the Becker–Döring equations). Asymptotics for the continuous model without diffusion is studied in [1, 17, 23, 24], whereas the long-time behaviour for continuous coagulation-fragmentation models taking into account diffusion is investigated in [16]. Note that all of the just-cited papers consider neither the possibility of shattering nor the existence of a maximal droplet mass, so that there is also no scattering. In this paper we include both of these processes. Inspired by the work of [16], we prove in § 2 that the solutions converge (with respect to the L_1 topology) towards the unique equilibrium with the same mass as the initial distribution. Moreover, in § 3 we derive stability of theses equilibria in a suitable topology.

2. Trend to equilibrium

In the sequel, we put $L_1 := L_1((0, y_0])$ and denote by $|\cdot|_1$ the norm of L_1. The closed subset of L_1, consisting of all $u \in L_1$ that are non-negative almost everywhere, is denoted by L_1^+. Furthermore, $L_{1,\mathrm{w}}$ stands for the space L_1 endowed with its weak topology.

Throughout this paper we assume that the following hypotheses are satisfied.

(H$_1$) $\varphi : L_1 \to (0, \infty)$ is uniformly Lipschitz continuous on bounded sets, weakly sequentially continuous and bounded.

(H$_2$) γ is a measurable function from $\Delta := \{(y, y');\ 0 < y' < y \leqslant y_0\}$ into $\mathbb{R}^+$, satisfying (1.1), and there exists $m_\gamma > 0$ with

$$\int_0^y \gamma(y, y')\,\mathrm{d}y' \leqslant m_\gamma \quad \text{a.a. } y \in (0, y_0].$$

(H$_3$) β_{c} is a measurable function from Δ into $\mathbb{R}^+$ satisfying (1.3) and

$$\int_0^{y+y'} \beta_{\mathrm{c}}(y + y', y'')\,\mathrm{d}y'' = 2 \tag{2.1}$$

for a.a. $(y, y') \in (0, y_0]^2$ with $y + y' \in (0, y_0]$.

(H$_4$) β_{s} is a measurable function from $\{(y, y');\ 0 < y - y_0 \leqslant y' \leqslant y_0\}$ into $\mathbb{R}^+$ satisfying (1.4) and

$$\int_{y+y'-y_0}^{y_0} \beta_{\mathrm{s}}(y + y', y'')\,\mathrm{d}y'' = 2 \tag{2.2}$$

for a.a. $(y, y') \in (0, y_0]^2$ with $y + y' \in (y_0, 2y_0]$.

(H$_5$) $P, Q, K \in L_\infty((0, y_0]^2, \mathbb{R}^+)$ are symmetric and P, Q satisfy (1.2), whereas $PK > 0$ a.e.

(H$_6$) There exists $H \in L_1^+$ with ess inf $H > 0$ and

(i) for $0 < y + y' < y_0$,

$$\gamma(y + y', y)H(y + y') = P(y, y')K(y, y')H(y)H(y');$$

(ii) for $0 < y + y',\ y + y'' < y_0$,

$$\begin{aligned}\beta_{\mathrm{c}}(y, y')Q(y'', y - y'')K(y'', y - y'')H(y'')H(y - y'')\\ = \beta_{\mathrm{c}}(y, y'')Q(y', y - y')K(y', y - y')H(y')H(y - y');\end{aligned}$$

(iii) for $0 < y - y_0 < y'$, $y'' < y_0$,

$$\begin{aligned}\beta_{\mathrm{s}}(y, y')K(y'', y - y'')H(y'')H(y - y'')\\ = \beta_{\mathrm{s}}(y, y'')K(y', y - y')H(y')H(y - y').\end{aligned}$$

We refer to examples 2.12–2.15 for kernels satisfying the hypotheses above. Equalities (2.1) and (2.2) reflect binary breakage in the shattering and scattering processes, respectively. Observe that, in combination with (1.3) and (1.4), they additionally imply that

$$\int_0^{y+y'} y''\beta_{\mathrm{c}}(y + y', y'')\,\mathrm{d}y'' = y + y' \tag{2.3}$$

for a.a. $(y, y') \in (0, y_0]^2$ with $y + y' \in (0, y_0]$ and

$$\int_{y+y'-y_0}^{y_0} y''\beta_{\mathrm{s}}(y + y', y'')\,\mathrm{d}y'' = y + y' \tag{2.4}$$

for a.a. $(y, y') \in (0, y_0]^2$ with $y + y' \in (y_0, 2y_0]$. In other words, shattering and scattering are mass-preserving processes.

Before making use of hypothesis (H_6), let us collect some already proven facts on global existence of solutions to problem ($*$), that is, for the ordinary differential equation

$$\left.\begin{aligned}\dot{u} &= \varphi(u)L[u], \quad t > 0,\\ u(0) &= u^0,\end{aligned}\right\} \tag{$**$}$$

considered in L_1.

Theorem 2.1. *Suppose that hypotheses (H_1)–(H_5) are satisfied. Then, given any $u^0 \in L_1^+$, problem ($**$) admits a unique solution $u(\,\cdot\,; u^0) \in C^1(\mathbb{R}^+, L_1)$, which, in addition, is non-negative and preserves the total mass, i.e.*

$$\int_0^{y_0} yu(t; u^0)(y)\,\mathrm{d}y = \int_0^{y_0} yu^0(y)\,\mathrm{d}y, \quad t \geqslant 0.$$

Moreover, the map $(t, u^0) \mapsto u(t; u^0)$ defines a semiflow on L_1^+.

Proof. This follows by an obvious modification of the proofs in [27] (there, the case $Q \equiv 1 - P$ is treated). A detailed proof is also given in [28]. □

In the following, we denote by $u := u(\,\cdot\,; u^0) \in C^1(\mathbb{R}^+, L_1)$ the unique solution to ($**$), and we write

$$u(t, y) := u(t; u^0)(y) \quad \text{a.a. } y \in (0, y_0], \quad t \geqslant 0,$$

if the initial value $u^0 \in L_1^+$ is fixed. Sometimes we suppress any of the variables t and y in a given formula. Furthermore, c or $c(u^0)$ will denote various constants, which may differ from occurrence to occurrence, but which are always independent of the free variables.

It is an easy consequence of hypothesis (H_6) that the function $u_\alpha \in L_1^+$, given by

$$u_\alpha(y) := H(y)\mathrm{e}^{\alpha y} \quad \text{a.a. } y \in (0, y_0], \tag{2.5}$$

is, for each $\alpha \in \mathbb{R}$, an equilibrium of problem $(**)$. Let us then introduce the map $V : L_1^+ \to \mathbb{R}^+ \cup \{\infty\}$ according to

$$V(v) := \int_0^{y_0} \left\{ v(y)\left[\log \frac{v(y)}{H(y)} - 1\right] + H(y) \right\} \mathrm{d}y, \quad v \in L_1^+,$$

which will turn out to be a Lyapunov function for $(**)$. Note that Fatou's lemma entails that V is sequentially lower semi-continuous. Hence V is weakly sequentially lower semi-continuous due to its convexity (see [12, proposition 2.3]).

Next, the proof of [16, lemma 3.1] can easily be modified to yield the following lemma, which will guarantee that the orbit of the motion through $u^0 \in L_1^+$ is relatively weakly compact in L_1, provided $V(u^0) < \infty$.

Lemma 2.2. *Let $w \in L_1^+$ be such that $V(w) < \infty$. Then, for each $\alpha \geqslant \mathrm{e}^2$ and each measurable subset A of $(0, y_0]$, we have*

$$\int_A w(y)\,\mathrm{d}y \leqslant 2\alpha \int_A H(y)\,\mathrm{d}y + \frac{2}{\log \alpha} V(w).$$

Furthermore, it is not difficult to adapt the ideas of [16, lemma C.1] in order to proof the next result. We refrain from giving details and refer to [28, lemma 3.9].

Lemma 2.3. *Suppose that $w \in L_1^+$ satisfies*

$$\gamma(y + y', y)w(y + y') = P(y, y')K(y, y')w(y)w(y') \tag{2.6}$$

for a.a. $(y, y') \in (0, y_0]^2$ with $0 < y + y' \leqslant y_0$. Then either $w = 0$ a.e. or there exists $\alpha \in \mathbb{R}$ such that $w(y) = H(y)\mathrm{e}^{\alpha y}$ for a.a. $y \in (0, y_0]$.

The main ingredient for examining large-time behaviour of the solutions consists of proving that V is a Lyapunov function for $(**)$, that is, that V is decreasing along orbits. Such a result will make heavy use of hypothesis (H_6) as well as of formulae (2.1) and (2.2). In order to carry through rigorously the technical details, we need an upper and lower bound for the solutions. This may be obtained by approximating the solution to $(**)$ by solutions to a modified problem, where the initial value and the kernels are truncated in a suitable way, paying attention to the detailed balance condition (H_6). But then these truncated kernels no longer obey equalities of type (2.1) and (2.2). Hence we also have to alter the reaction terms slightly in order to guarantee that V is still decreasing along orbits of solutions to the modified problem. For that purpose, let us introduce some further notations. Define the set

$$\mathcal{E} := \{(y, y') \in (0, y_0]^2;\ y + y' < y_0\},$$

and, for $n \geqslant 1$, the sets

$$\begin{aligned}
A_n &:= \{(y, y') \in \mathcal{E};\ \gamma(y + y', y) \leqslant n\},\\
B_n &:= \{(y, y') \in \mathcal{E};\ \beta_{\mathrm{c}}(y + y', y) \leqslant n\},\\
C_n &:= \{(y, y') \in (0, y_0]^2 \setminus \mathcal{E};\ \beta_{\mathrm{s}}(y + y', y) \leqslant n\},
\end{aligned}$$

and observe that (y, y') belongs to any one of the sets A_n, B_n or C_n if and only if (y', y) does. Furthermore, truncate the kernels according to

$$\gamma_n(y+y',y) := \begin{cases} \gamma(y+y',y), & (y,y') \in A_n \cap B_n, \\ 0 & \text{otherwise}, \end{cases}$$

$$\beta_{\mathrm{c},n}(y+y',y) := \begin{cases} \beta_{\mathrm{c}}(y+y',y), & (y,y') \in A_n \cap B_n, \\ 0 & \text{otherwise}, \end{cases}$$

$$\beta_{\mathrm{s},n}(y+y',y) := \begin{cases} \beta_{\mathrm{s}}(y+y',y), & (y,y') \in C_n, \\ 0 & \text{otherwise}, \end{cases}$$

$$K_n(y,y') := \begin{cases} K(y,y'), & (y,y') \in (A_n \cap B_n) \cup C_n, \\ 0 & \text{otherwise}. \end{cases}$$

Then K_n is symmetric and γ_n, $\beta_{\mathrm{c},n}$ and $\beta_{\mathrm{s},n}$ satisfy hypotheses (H_2), (H_3) and (H_4), respectively. Furthermore,

$$\gamma_n \nearrow \gamma, \qquad \beta_{\mathrm{c},n} \nearrow \beta_{\mathrm{c}}, \qquad \beta_{\mathrm{s},n} \nearrow \beta_{\mathrm{s}}, \quad K_n \nearrow K \tag{2.7}$$

pointwise on the domains of γ, β_{c}, β_{s} and K. Finally, the truncated kernels satisfy the detailed balance condition (H_6) with the same function H and the same probabilities P and Q.

In addition, define for $w \in L_1$ and a.a. $y \in (0, y_0]$

$$\begin{aligned}
L_{\mathrm{b},n}[w](y) &:= \int_y^{y_0} \gamma_n(y',y) w(y')\,\mathrm{d}y' - \tfrac{1}{2} w(y) \int_0^y \gamma_n(y,y')\,\mathrm{d}y', \\
L_{\mathrm{c},n}[w](y) &:= \frac{1}{2}\int_0^y K_n(y', y-y') P(y', y-y') w(y') w(y-y')\,\mathrm{d}y' \\
&\quad + \frac{1}{2}\int_y^{y_0}\int_0^{y'} K_n(y'', y'-y'') Q(y'', y'-y'') \\
&\qquad\qquad \times \beta_{\mathrm{c},n}(y',y) w(y'') w(y'-y'')\,\mathrm{d}y''\mathrm{d}y' \\
&\quad - w(y)\int_0^{y_0-y} K_n(y,y') P(y,y') w(y')\,\mathrm{d}y' \\
&\quad - \tfrac{1}{2} w(y)\int_0^{y_0-y}\int_0^{y+y'} \beta_{\mathrm{c},n}(y+y', y'')\,\mathrm{d}y'' K_n(y,y') Q(y,y') w(y')\,\mathrm{d}y', \\
L_{\mathrm{s},n}[w](y) &:= \frac{1}{2}\int_{y_0}^{y_0+y}\int_{y'-y_0}^{y_0} \beta_{\mathrm{s},n}(y',y) K_n(y'', y'-y'') w(y'') w(y'-y'')\,\mathrm{d}y''\mathrm{d}y' \\
&\quad - \frac{1}{2} w(y) \int_{y_0-y}^{y_0}\int_{y+y'-y_0}^{y_0} \beta_{\mathrm{s},n}(y+y', y'')\,\mathrm{d}y'' K_n(y,y') w(y')\,\mathrm{d}y'
\end{aligned}$$

and, furthermore,

$$L_n[w] := L_{\mathrm{b},n}[w] + L_{\mathrm{c},n}[w] + L_{\mathrm{s},n}[w], \quad w \in L_1.$$

In the sequel, we denote by $|\cdot|_\infty$ the norm of $L_\infty := L_\infty((0, y_0])$.

LEMMA 2.4. *Given $n \geqslant 1$ and any non-negative $w^0 \in L_\infty$, there exists a unique solution $w := w(\,\cdot\,; w^0) \in C^1(\mathbb{R}^+, L_\infty)$ for the problem*

$$\begin{aligned} \dot{w} &= \varphi(w) L_n[w], \quad t > 0, \\ w(0) &= w^0. \end{aligned}$$

Moreover, this solution is non-negative and, in addition, if $w^0 \geqslant r_0$ a.e. for some $r_0 \in (0, \infty)$, then, for any $T > 0$, there exists $r_T > 0$ such that

$$w(t) \geqslant r_T \quad \text{a.e.}, \quad 0 \leqslant t \leqslant T. \tag{2.8}$$

Proof. According to hypotheses (H_1)–(H_5), we have

$$|\varphi(w) L_n[w]|_\infty \leqslant c(1 + |w|_1)|w|_\infty, \quad w \in L_\infty. \tag{2.9}$$

From this, existence of a unique solution $w \in C^1(J(w^0), L_\infty)$ follows, where $J(w^0)$ denotes the maximal interval of existence. That this solution is non-negative may be obtained along the lines of the proof of [27, theorem 2.4]. Observe then that

$$\int_0^{y_0} L_{\mathrm{b},n}[v](y)\,\mathrm{d}y \leqslant c|v|_1, \qquad \int_0^{y_0} L_{\mathrm{c},n}[v](y)\,\mathrm{d}y \leqslant 0, \qquad \int_0^{y_0} L_{\mathrm{s},n}[v](y)\,\mathrm{d}y = 0$$

for $v \in L_1^+$. Since $w(t) \in L_1^+$ for $t \in J(w^0)$, Gronwall's inequality applies to provide $c := c(w^0)$ with

$$|w(t)|_1 \leqslant c\mathrm{e}^{ct}, \quad t \in J(w^0),$$

so that (2.9) entails $J(w^0) = \mathbb{R}^+$. Finally, it remains to prove (2.8). Fix $T > 0$ arbitrarily and put

$$\omega := \|\varphi\|_\infty \Big(m_\gamma + \|K\|_\infty \max_{0 \leqslant t \leqslant T} |w(t)|_1 \Big).$$

Since $w(s) \geqslant 0$ a.e., we deduce for $0 \leqslant t \leqslant T$,

$$\begin{aligned} w(t) &= \mathrm{e}^{-\omega t} w^0 + \int_0^t \mathrm{e}^{-\omega(t-s)} \{\varphi(w(s)) L_n[w(s)] + \omega w(s)\}\,\mathrm{d}s \\ &\geqslant \mathrm{e}^{-\omega T} r_0 =: r_T \quad \text{a.e.} \end{aligned}$$

□

We also need the following lemma, whose prove can be found in [16, lemma A.2].

LEMMA 2.5. *Let $\Omega \subset \mathbb{R}^m$, $m \geqslant 1$, be a measurable and bounded set. Assume that $h_n, h \in L_\infty(\Omega)$ are such that $\|h_n\|_\infty \leqslant c$ for $n \geqslant 1$ and $h_n \to h$ a.e. Then, provided that $v_n \to v$ in $L_{1,\mathrm{w}}(\Omega)$, we have $h_n v_n \to hv$ in $L_{1,\mathrm{w}}(\Omega)$.*

The next lemma will ensure, in particular, that the solutions to the modified problem, being provided by lemma 2.4, indeed approximate the original solution $u(\,\cdot\,; u^0)$.

LEMMA 2.6. *Assume that $w_n \to w$ in $L_{1,\mathrm{w}}$.*

(i) *Defining, for* $(y, y') \in \mathcal{E}$,

$$v_n(y, y') := \gamma_n(y + y', y)w_n(y + y')$$

and

$$v(y, y') := \gamma(y + y', y)w(y + y'),$$

we have $v_n \to v$ *in* $L_{1,\mathrm{w}}(\mathcal{E})$.

(ii) *Defining, for* $(y, y') \in \mathcal{E}$,

$$z_n(y, y') := P(y, y')K_n(y, y')w_n(y)w_n(y')$$

and

$$z(y, y') := P(y, y')K(y, y')w(y)w(y'),$$

we have $z_n \to z$ *in* $L_{1,\mathrm{w}}(\mathcal{E})$.

(iii) $L_n[w_n] \to L[w]$ *in* $L_{1,\mathrm{w}}$.

Proof. Given $f \in L_\infty(\mathcal{E})$, use Fubini's theorem to deduce

$$\left| \int_{\mathcal{E}} f(y, y')[v_n(y, y') - v(y, y')] \, \mathrm{d}(y, y') \right| \\ \leqslant \|f\|_\infty \int_0^{y_0} a_n(y)|w(y)| \, \mathrm{d}y + \left| \int_0^{y_0} h_n(y)[w(y) - w_n(y)] \, \mathrm{d}y \right|, \quad (2.10)$$

where

$$a_n(y) := \int_0^y |\gamma_n(y, y') - \gamma(y, y')| \, \mathrm{d}y', \qquad h_n(y) := \int_0^y f(y', y - y')\gamma_n(y, y') \, \mathrm{d}y'.$$

Due to hypothesis (H_2) and (2.7), an application of Lebesgue's theorem yields that the first term on the right-hand side of (2.10) converges to 0 as $n \to \infty$. Next, observe that, for a.a. $y \in (0, y_0]$, we have, by virtue of Fubini's theorem,

$$f(\cdot, y - \cdot) \in L_\infty((0, y)), \quad \text{with } \|f(\cdot, y - \cdot)\|_{L_\infty((0,y))} \leqslant \|f\|_\infty.$$

We obtain $|h_n|_\infty \leqslant \|f\|_\infty m_\gamma$ and, using Lebesgue's theorem,

$$h_n(y) \to h(y) := \int_0^y f(y', y - y')\gamma(y, y') \, \mathrm{d}y' \quad \text{a.a. } y \in (0, y_0],$$

where $h \in L_\infty$. Lemma 2.5 entails now that $v_n \to v$ in $L_{1,\mathrm{w}}(\mathcal{E})$. All other statements can be proven in a similar way (for (iii), recall (2.1) and (2.2)). Therefore, we refrain from giving more details and refer to [28]. □

Let us introduce some further notations. Define the map $\mathcal{J} : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{\infty\}$ by

$$\mathcal{J}(a, b) := \begin{cases} (a - b)(\log a - \log b), & a, b > 0, \\ 0, & a = b = 0, \\ \infty & \text{otherwise.} \end{cases}$$

In order to shorten the formulae, we agree upon putting

$$y''' \equiv y + y' - y'', \quad 0 < y'' < y + y'.$$

Moreover, we set, for $v \in L_1^+$,

$$\begin{aligned}
D(v) &:= \frac{1}{2}\int_{\mathcal{E}} \mathcal{J}(P(y,y')K(y,y')v(y)v(y'), \gamma(y+y',y)v(y+y'))\,\mathrm{d}(y,y'),\\
F(v) &:= \frac{1}{8}\int_{\mathcal{W}} \mathcal{J}(\beta_{\mathrm{c}}(y+y',y)Q(y'',y''')K(y'',y''')v(y'')v(y'''),\\
&\qquad\qquad \beta_{\mathrm{c}}(y+y',y'')Q(y,y')K(y,y')v(y)v(y'))\,\mathrm{d}(y,y',y''),\\
G(v) &:= \frac{1}{8}\int_{\mathcal{S}} \mathcal{J}(\beta_{\mathrm{s}}(y+y',y)K(y'',y''')v(y'')v(y'''),\\
&\qquad\qquad \beta_{\mathrm{s}}(y+y',y'')K(y,y')v(y)v(y'))\,\mathrm{d}(y,y',y''),
\end{aligned}$$

where the sets $\mathcal{W}$ and $\mathcal{S}$ are given by

$$\begin{aligned}
\mathcal{W} &:= \{(y,y',y'') \in (0,y_0]^3;\ y'' < y + y' < y_0\},\\
\mathcal{S} &:= \{(y,y',y'') \in (0,y_0]^3;\ y_0 - y'' < y + y' - y'' < y_0\}.
\end{aligned}$$

Finally, we define $D_n(v)$, $F_n(v)$ and $G_n(v)$ analogously, but with $(\gamma_n, \beta_{\mathrm{c},n}, \beta_{\mathrm{s},n}, K_n)$ instead of $(\gamma, \beta_{\mathrm{c}}, \beta_{\mathrm{s}}, K)$.

Now we are in position to prove that V is indeed a Lyapunov function for $(**)$.

PROPOSITION 2.7. *Let $u^0 \in L_1^+$ be such that $V(u^0) < \infty$ and denote by $u = u(\,\cdot\,; u^0)$ the unique non-negative solution to $(**)$ in $C^1(\mathbb{R}^+, L_1)$. Then we have*

$$0 \leqslant V(u(t)) \leqslant V(u(s)) < \infty, \quad t \geqslant s \geqslant 0, \tag{2.11}$$

and

$$[t \mapsto \varphi(u(t))D(u(t))] \in L_1(\mathbb{R}^+). \tag{2.12}$$

Proof. For $n \geqslant 1$, set

$$u_n^0(y) := \min\{n, \max\{u^0(y), H(y)/n\}\} \quad \text{a.a. } y \in (0, y_0],$$

and observe that $0 < \min\{n, (1/n)\operatorname{ess\,inf} H\} \leqslant u_n^0 \leqslant n$ a.e. and $u_n^0 \to u^0$ in L_1. Further, we have

$$\int_0^{y_0} u_n^0 \log\frac{u_n^0}{H}\,\mathrm{d}y \leqslant \biggl(\int_{S_n} + \int_{T_n}\biggr) u^0 \log\frac{u^0}{H}\,\mathrm{d}y, \quad n \geqslant 1,$$

where we put

$$S_n := \left[\frac{H}{n} \leqslant u^0 < n\right] \quad \text{and} \quad T_n := [H < n \leqslant u^0].$$

Taking into account the fact that $V(u^0) < \infty$ and $r|\log r| \leqslant r\log r + 2/\mathrm{e}$, $r \geqslant 0$, imply $u^0 \log(u^0/H) \in L_1$, Lebesgue's theorem yields

$$\limsup_n V(u_n^0) \leqslant V(u^0). \tag{2.13}$$

Next, lemma 2.4 entails the existence of a solution $u_n := u_n(\,\cdot\,; u_n^0) \in C^1(\mathbb{R}^+, L_\infty)$ to the problem

$$\begin{aligned} \dot{w} &= \varphi(w) L_n[w], \quad t > 0, \\ w(0) &= u_n^0 \end{aligned}$$

satisfying, for each $T > 0$,

$$0 < r_n^1(T) \leqslant u_n(t) \leqslant r_n^2(T) < \infty \quad \text{a.e.,} \quad 0 \leqslant t \leqslant T, \tag{2.14}$$

for some constants $r_n^j(T)$. This enables us to deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} V(u_n(t)) = \varphi(u_n(t)) \int_0^{y_0} \log \frac{u_n(t,y)}{H(y)} L_n[u_n(t)](y)\,\mathrm{d}y \tag{2.15}$$

for $n \geqslant 1$ and $0 \leqslant t \leqslant T$. Note that Fubini's theorem applies throughout in the following because of (2.14). Little effort then yields

$$\int_0^{y_0} \log \frac{u_n(t,y)}{H(y)} \{L_{\mathrm{b},n}[u_n(t)](y) + L_{\mathrm{c},n}^{(P)}[u_n(t)](y)\}\,\mathrm{d}y = -D_n(u_n(t)), \tag{2.16}$$

for $n \geqslant 1$ and $0 \leqslant t \leqslant T$, where $L_{\mathrm{c},n}^{(P)}$ consists of those integral terms of $L_{\mathrm{c},n}$ involving P but not Q. Furthermore, we compute

$$\begin{aligned} &\int_0^{y_0} \log \frac{u_n(y)}{H(y)} L_{\mathrm{s},n}[u_n](y)\,\mathrm{d}y \\ &\quad = \frac{1}{2} \int_{\mathcal{S}} \left\{ \log \frac{u_n(y'')}{H(y'')} - \log \frac{u_n(y)}{H(y)} \right\} \\ &\qquad\qquad \times \beta_{\mathrm{s},n}(y+y', y'') K_n(y,y') u_n(y) u_n(y')\,\mathrm{d}(y,y',y'') \\ &\quad = \frac{1}{4} \int_{\mathcal{S}} \left\{ \log \frac{u_n(y'') u_n(y''')}{H(y'') H(y''')} - \log \frac{u_n(y) u_n(y')}{H(y) H(y')} \right\} \\ &\qquad\qquad \times \beta_{\mathrm{s},n}(y+y', y'') K_n(y,y') u_n(y) u_n(y')\,\mathrm{d}(y,y',y''), \end{aligned}$$

where we have taken into account the symmetry of K_n and the fact that $\beta_{\mathrm{s},n}$ satisfies (1.4). The transformation $\mathcal{S} \to \mathcal{S}, (y, y', y'') \mapsto (y'', y''', y)$ entails then that the right-hand side of the above equality coincides with

$$\begin{aligned} &\frac{1}{8} \int_{\mathcal{S}} \left\{ \log \frac{u_n(y'') u_n(y''')}{H(y'') H(y''')} - \log \frac{u_n(y) u_n(y')}{H(y) H(y')} \right\} \\ &\qquad\qquad \times \beta_{\mathrm{s},n}(y+y', y'') K_n(y,y') u_n(y) u_n(y')\,\mathrm{d}(y,y',y'') \\ &\quad + \frac{1}{8} \int_{\mathcal{S}} \left\{ \log \frac{u_n(y) u_n(y')}{H(y) H(y')} - \log \frac{u_n(y'') u_n(y''')}{H(y'') H(y''')} \right\} \\ &\qquad\qquad \times \beta_{\mathrm{s},n}(y+y', y) K_n(y'',y''') u_n(y'') u_n(y''')\,\mathrm{d}(y,y',y''). \end{aligned}$$

Finally, due to hypothesis (H_6), we may rewrite this last expression to get

$$\int_0^{y_0} \log \frac{u_n(t,y)}{H(y)} L_{\mathrm{s},n}[u_n(t)](y)\,\mathrm{d}y = -G_n(u_n(t)), \tag{2.17}$$

for $n \geqslant 1$ and $0 \leqslant t \leqslant T$. Likewise, one derives

$$\int_0^{y_0} \log \frac{u_n(t,y)}{H(y)} L_{\mathrm{c},n}^{(Q)}[u_n(t)](y)\,\mathrm{d}y = -F_n(u_n(t)), \tag{2.18}$$

where $L_{\mathrm{c},n}^{(Q)}$ are those integral terms of $L_{\mathrm{c},n}$ involving Q but not P. Therefore, equations (2.15)–(2.18), in combination with (2.13), yield, for $n \geqslant 1$ and $0 \leqslant t \leqslant T$,

$$\begin{aligned} V(u_n(t)) + \int_0^t \varphi(u_n(\sigma))\{D_n(u_n(\sigma)) + F_n(u_n(\sigma)) + G_n(u_n(\sigma))\}\,\mathrm{d}\sigma &= V(u_n^0) \\ &\leqslant c(u^0) \\ &< \infty. \end{aligned} \tag{2.19}$$

Consequently,

$$V(u_n(t)) \leqslant c(u^0), \quad n \geqslant 1, \quad t \geqslant 0, \tag{2.20}$$

since each of the terms $D_n(u_n(\sigma))$, $F_n(u_n(\sigma))$ and $G_n(u_n(\sigma))$ is non-negative. Hence lemma 2.2 leads to

$$|u_n(t)|_1 \leqslant c(u^0), \quad n \geqslant 1, \quad t \geqslant 0, \tag{2.21}$$

and, invoking additionally the Dunford–Pettis theorem [11, theorem 4.21.2], we see that the set $\{u_n(t); n \geqslant 1\}$ is relatively weakly compact in L_1 for each $t \geqslant 0$. Next, from (2.7) and hypotheses (H_1)–(H_5), we derive

$$|\varphi(v)L_n[v]|_1 \leqslant c(1 + |v|_1)|v|_1, \quad v \in L_1, \quad n \geqslant 1, \tag{2.22}$$

with c being independent of $n \geqslant 1$. This and (2.21) imply

$$|u_n(t) - u_n(s)|_1 \leqslant c(u^0)|t - s|, \quad t, s \geqslant 0, \quad n \geqslant 1. \tag{2.23}$$

In particular, the set $\{u_n; n \geqslant 1\}$ is equicontinuous with respect to the weak topology of L_1. Now fix $T > 0$ arbitrarily. Then the Arzelà–Ascoli theorem [26, theorem 1.3.2] entails that there exist $\bar{u} \in C([0,T], L_{1,\mathrm{w}})$ and a subsequence (n') such that

$$u_{n'} \to \bar{u} \quad \text{in } C([0,T], L_{1,\mathrm{w}}). \tag{2.24}$$

Clearly, $\bar{u}$ belongs to $C^{1-}([0,T], L_1)$ due to (2.23), that is, $\bar{u}$ is Lipschitz continuous with respect to the L_1 topology. Furthermore, thanks to lemma 2.6, we have

$$L_{n'}[u_{n'}(\sigma)] \to L[\bar{u}(\sigma)] \quad \text{in } L_{1,\mathrm{w}}, \quad 0 \leqslant \sigma \leqslant T. \tag{2.25}$$

Since φ is weakly sequentially continuous, an application of Lebesgue's theorem and equations (2.21), (2.22), (2.24) and (2.25) yield

$$\int_0^t \varphi(u_{n'}(\sigma))L_{n'}[u_{n'}(\sigma)]\,\mathrm{d}\sigma \to \int_0^t \varphi(\bar{u}(\sigma))L[\bar{u}(\sigma)]\,\mathrm{d}\sigma \quad \text{in } L_{1,\mathrm{w}}, \quad 0 \leqslant t \leqslant T,$$

so that a renewed use of (2.24) shows that

$$\bar{u}(t) = u^0 + \int_0^t \varphi(\bar{u}(\sigma))L[\bar{u}(\sigma)]\,\mathrm{d}\sigma, \quad 0 \leqslant t \leqslant T.$$

Hence $\bar{u} = u(\,\cdot\,; u^0)|_{[0,T]}$ due to uniqueness of solutions to $(**)$. Consequently, we have

$$u_{n'} \to u(\,\cdot\,; u^0) \quad \text{in } C([0,T], L_{1,\mathrm{w}}). \tag{2.26}$$

Since V is weakly lower semi-continuous and since $T > 0$ was arbitrary, we deduce from (2.19) and (2.13) that (2.11) is indeed true for $t \geqslant s = 0$. The semiflow property then yields the general case of (2.11).

Hence it remains to prove (2.12). According to (2.26), we may apply lemma 2.6 to obtain

$$\begin{aligned} \gamma_{n'}(y+y',y)u_{n'}(\sigma,y+y') &\to \gamma(y+y',y)u(\sigma,y+y') && \text{in } L_{1,\mathrm{w}}(\mathcal{E}),\\ P(y,y')K_{n'}(y,y')u_{n'}(\sigma,y)u_{n'}(\sigma,y') &\to P(y,y')K(y,y')u(\sigma,y)u(\sigma,y') && \text{in } L_{1,\mathrm{w}}(\mathcal{E}), \end{aligned}$$

for $0 \leqslant \sigma \leqslant T$. Since the function $\mathcal{J}$, appearing in the definition of $D(v)$, is convex and lower semi-continuous, we obtain from the above convergence, from Fatou's lemma and from (2.19), that

$$\int_0^T \varphi(u(\sigma))D(u(\sigma))\,\mathrm{d}\sigma \leqslant \liminf_{n'} \int_0^T \varphi(u_{n'}(\sigma))D_{n'}(u_{n'}(\sigma))\,\mathrm{d}\sigma \leqslant c(u^0),$$

whereby $c(u^0)$ does not depend on $T > 0$. □

Recall that the equilibria $u_\alpha, \alpha \in \mathbb{R}$ are given by (2.5). Clearly, given any $\varrho > 0$, there exists $\alpha(\varrho) \in \mathbb{R}$ uniquely such that $M(u_{\alpha(\varrho)}) = \varrho$, where the mass $M(v)$ of $v \in L_1^+$ is defined as

$$M(v) := \int_0^{y_0} y v(y)\,\mathrm{d}y.$$

Now we can state the result concerning convergence towards equilibrium.

THEOREM 2.8. *Given $u^0 \in L_1^+ \setminus \{0\}$ with $V(u^0) < \infty$, choose $\alpha \in \mathbb{R}$ such that $M(u_\alpha) = M(u^0)$. Then, given any sequence $t_n \nearrow \infty$ and any $T > 0$, the solution $u = u(\,\cdot\,; u^0)$ to problem $(**)$ satisfies*

$$u(\cdot + t_n; u^0) \to u_\alpha \quad \textit{in } C([0,T], L_{1,\mathrm{w}}). \tag{2.27}$$

In addition, if there exists $r \in L_1^+$ such that, for a.a. $y \in (0, y_0)$,

$$\gamma(\cdot, y) \leqslant r(y) \quad \textit{a.e. on } (y, y_0), \tag{2.28}$$

and if $u^0 > 0$ a.e., then

$$u(\cdot + t_n; u^0) \to u_\alpha \quad \textit{in } C([0,T], L_1). \tag{2.29}$$

Proof. Put

$$u_n(t) := u(t + t_n; u^0) = u(t; u(t_n; u^0)), \quad t \geqslant 0, \quad n \geqslant 1,$$

so that, according to proposition 2.7,

$$V(u_n(t)) \leqslant V(u^0), \quad t \geqslant 0, \quad n \geqslant 1. \tag{2.30}$$

Analogously to the proof of proposition 2.7, we deduce the existence of a function $\bar{u} \in C^{1-}([0,T], L_1)$ and of a subsequence (n') such that $u_{n'} \to \bar{u}$ in $C([0,T], L_{1,\mathrm{w}})$.

Obviously, we have $\bar{u}(t) \in L_1^+$ for $0 \leqslant t \leqslant T$. Furthermore, as in the proof of proposition 2.7, we infer

$$0 \leqslant \int_0^T \varphi(\bar{u}(t))D(\bar{u}(t))\,\mathrm{d}t \leqslant \liminf_{n'} \int_0^T \varphi(u_{n'}(t))D(u_{n'}(t))\,\mathrm{d}t.$$

Thanks to (2.12), the latter expression equals zero. Therefore, $D(\bar{u}(t)) = 0$ for a.a. $0 \leqslant t \leqslant T$ since φ has no zeros. By definition of D, lemma 2.3 entails that $\bar{u}(t)$ is an equilibrium of the form (2.5) for a.a. $t \in [0, T]$. But since

$$M(\bar{u}(t)) = M(u_{n'}(t)) = M(u^0) = M(u_\alpha), \quad 0 \leqslant t \leqslant T,$$

according to theorem 2.1, we deduce that $\bar{u}$ is independent of time due to continuity, and it coincides with u_α. Therefore, $u_{n'} \to u_\alpha$ in $C([0,T], L_{1,\mathrm{w}})$, which leads to (2.27), since the limit does not depend on the extracted subsequence.

Let (2.28) be true, so that (2.27) implies, for $T > 0$,

$$L_{\mathrm{b}}^1[u_n(t)](y) \to L_{\mathrm{b}}^1[u_\alpha](y) \quad \text{a.a. } y \in (0, y_0], \quad 0 \leqslant t \leqslant T,$$

where we put

$$L_{\mathrm{b}}^1[v](y) := \int_y^{y_0} \gamma(y', y)v(y')\,\mathrm{d}y', \quad \text{a.a. } y \in (0, y_0], \quad v \in L_1.$$

Moreover, invoking (2.30) and lemma 2.2, we get

$$|L_{\mathrm{b}}^1[u_n(t)](y)| \leqslant |u_n(t)|_1 r(y) \leqslant c(u^0) r(y) \quad \text{a.a. } y \in (0, y_0], \quad 0 \leqslant t \leqslant T, \tag{2.31}$$

with $c(u^0) > 0$ depending neither on $n \geqslant 1$ nor on $t \in [0, T]$. Thus Lebesgue's theorem and (2.27) entail

$$\varphi(u_n) L_{\mathrm{b}}^1[u_n] \to \varphi(u_\alpha) L_{\mathrm{b}}^1[u_\alpha] \quad \text{in } L_1((0,T) \times (0, y_0]), \tag{2.32}$$

since φ is weakly sequentially continuous and bounded. For $v \in L_1$, set

$$h(v)(y) := \int_0^{y_0 - y} P(y, y')K(y, y')v(y')\,\mathrm{d}y' \quad \text{a.a. } y \in (0, y_0].$$

Analogously as above, we then have

$$\varphi(u_n)h(u_n) \to \varphi(u_\alpha)h(u_\alpha) \quad \text{in } L_1((0,T) \times (0, y_0]). \tag{2.33}$$

Next, we take up the idea of the proof of lemma 2.4 in order to deduce that $u^0 > 0$ a.e. implies $u(t; u^0) > 0$ a.e. for each $t \geqslant 0$. Fix $\lambda > 1$ and observe that the inequality

$$|\eta - \xi| \leqslant (\lambda - 1)\xi + \frac{1}{\log \lambda}(\eta - \xi)(\log \eta - \log \xi), \quad \xi, \eta > 0,$$

holds, from which we derive

$$
\begin{aligned}
|\varphi(u_n)u_n h(u_n) - \varphi(u_n)L_{\mathrm{b}}^1[u_n]|_{L_1((0,T)\times(0,y_0])} \\
&\leqslant \int_0^T \varphi(u_n)\int_0^{y_0}\int_0^{y_0-y} |P(y,y')K(y,y')u_n(y)u_n(y') \\
&\qquad\qquad - \gamma(y+y',y)u_n(y+y')|\,\mathrm{d}y'\mathrm{d}y\mathrm{d}t \\
&\leqslant (\lambda-1)|\varphi(u_n)L_{\mathrm{b}}^1[u_n]|_{L_1((0,T)\times(0,y_0])} + \frac{2}{\log\lambda}\int_0^T \varphi(u_n)D(u_n)\,\mathrm{d}t.
\end{aligned}
$$

Taking the $\limsup_{n\nearrow\infty}$ on both sides and letting λ tend to 1, equations (2.12), (2.31) and (2.32) provide

$$
\varphi(u_n)u_n h(u_n) \to \varphi(u_\alpha)L_{\mathrm{b}}^1[u_\alpha] = \varphi(u_\alpha)u_\alpha h(u_\alpha) \quad \text{in } L_1((0,T)\times(0,y_0])
$$

as $n\nearrow\infty$, whereby the equality is implied by hypothesis (H_6). Therefore, recalling equation (2.33), we may extract a subsequence (n') such that $\varphi(u_{n'})u_{n'}h(u_{n'})$ and $\varphi(u_{n'})h(u_{n'})$ converge pointwise a.e. on $(0,T)\times(0,y_0]$ towards $\varphi(u_\alpha)u_\alpha h(u_\alpha)$ and $\varphi(u_\alpha)h(u_\alpha)$, respectively. But this implies that $u_{n'}\to u_\alpha$ a.e. on $(0,T)\times(0,y_0]$, since $\varphi(u_\alpha)h(u_\alpha)>0$ by virtue of hypotheses (H_1) and (H_5). Analogously to (2.21), the set $\{u_n(t);\ n\geqslant 1,\ 0\leqslant t\leqslant T\}$ is bounded in L_1, so that (2.27) gives $u_{n'}\to u_\alpha$ in $L_{1,\mathrm{w}}((0,T)\times(0,y_0])$ and a.e. on $(0,T)\times(0,y_0]$. Hence

$$
u_{n'} \to u_\alpha \quad \text{in } L_1((0,T)\times(0,y_0]), \tag{2.34}
$$

from which we derive (see [27, lemma 2.1])

$$
L[u_{n'}] \to L[u_\alpha] = 0 \quad \text{in } L_1((0,T)\times(0,y_0]). \tag{2.35}
$$

Observing

$$
u_{n'}(t) = u_{n'}(s) + \int_s^t \varphi(u_{n'}(\sigma))L[u_{n'}(\sigma)]\,\mathrm{d}\sigma, \quad 0\leqslant s\leqslant t,
$$

we then see that, for each $t\in(0,T]$,

$$
\begin{aligned}
t|u_{n'}(t)-u_\alpha|_1 &\leqslant |u_{n'}-u_\alpha|_{L_1((0,T)\times(0,y_0])} + \|\varphi\|_\infty \int_0^t\int_s^t |L[u_{n'}(\sigma)]|_1\,\mathrm{d}\sigma\mathrm{d}s \\
&\leqslant |u_{n'}-u_\alpha|_{L_1((0,T)\times(0,y_0])} + T\|\varphi\|_\infty |L[u_{n'}]|_{L_1((0,T)\times(0,y_0])}.
\end{aligned}
$$

Recalling (2.34) and (2.35), it therefore holds that $u_{n'}\to u_\alpha$ in $C((0,T],L_1)$. Assertion (2.29) is now evident. □

REMARK 2.9. Theorem 2.8 implies that there exist no further equilibria in L_1^+ for which V is finite.

REMARK 2.10. Note that it holds that $V(w)<\infty$ for $w\in L_p^+$, provided $p>1$. This is a consequence of the properties of H, Hölder's inequality and the fact that

$$
x|\log x| \leqslant c(\varepsilon)(x^{1+\varepsilon}+x^{1-\varepsilon}), \quad x>0, \quad \varepsilon>0.
$$

REMARK 2.11. Observe that the asymptotic distribution provided by theorem 2.8 depends merely on the total mass of the initial distribution and not on its shape, which seems to be consistent with numerical simulations and physical theory (for details, see [13, 15, 21, 25]).

It may be worthwhile to present some examples of kernels satisfying the imposed assumptions.

EXAMPLE 2.12. If φ is defined as in (1.5), then hypothesis (H_1) is satisfied, provided $\Phi : \mathbb{R}^2 \to (0,\infty)$ is uniformly Lipschitz continuous on bounded sets and bounded.

EXAMPLE 2.13. Let $P \in C((0,y_0]^2,(0,\infty))$ be symmetric and let $q \in C((0,y_0],\mathbb{R}^+)$ be such that

$$0 < P(y,y') + q(y+y') \leqslant 1, \quad 0 < y+y' \leqslant y_0.$$

Assume $\alpha \geqslant 0$ and $0 \geqslant \alpha-\beta > -1$ and define, for arbitrary constants $K^*, \gamma^* > 0$,

$$\begin{aligned}
Q(y,y') &:= q(y+y'), && 0 < y+y' \leqslant y_0,\\
K(y,y') &:= K^*(y+y')^{\alpha}, && 0 < y,y' \leqslant y_0,\\
\gamma(y,y') &:= \gamma^* P(y-y',y')y^{\beta}[y'(y-y')]^{\alpha-\beta}, && 0 < y' < y \leqslant y_0,\\
\beta_{\mathrm{c}}(y,y') &:= c_{\alpha,\beta} y^{-1-2\alpha+2\beta}[y'(y-y')]^{\alpha-\beta}, && 0 < y' < y \leqslant y_0,\\
\beta_{\mathrm{s}}(y,y') &:= f_{\mathrm{s}}(y)[y'(y-y')]^{\alpha-\beta}, && 0 < y-y_0 \leqslant y' \leqslant y_0,
\end{aligned}$$

where $c_{\alpha,\beta} := (B(\alpha-\beta+2,\alpha-\beta+1))^{-1}$, with B denoting the beta function, and where

$$f_{\mathrm{s}}(y) := y\biggl(\int_{y-y_0}^{y_0} y'[y'(y-y')]^{\alpha-\beta}\mathrm{d}y'\biggr)^{-1}, \quad y_0 < y < 2y_0.$$

Then hypotheses (H_2)–(H_6) are satisfied, with

$$H(y) := \frac{\gamma^*}{K^*}y^{\alpha-\beta}, \quad y \in (0,y_0].$$

Furthermore, inequality (2.28) holds provided $\alpha = \beta$.

EXAMPLE 2.14. Analogously as in [16], we may define

$$\begin{aligned}
K(y,y') &:= r\mathrm{e}^{-y^2-(y')^2}, && 0 < y,y' \leqslant y_0,\\
\gamma(y,y') &:= s\mathrm{e}^{-(y-2y')^2}, && 0 < y' < y \leqslant y_0,\\
\beta_{\mathrm{s}}(y,y') &:= f(y)\mathrm{e}^{-4y(y-y')}, && 0 < y-y_0 \leqslant y' \leqslant y_0,
\end{aligned}$$

for some $r, s > 0$, where

$$f(y) := y\biggl(\int_{y-y_0}^{y_0} y''\mathrm{e}^{-4y(y-y'')}\,\mathrm{d}y''\biggr)^{-1}, \quad y_0 < y < 2y_0.$$

Then, for $P \equiv 1$ and $Q \equiv 0$, hypotheses (H_2)–(H_6) hold, with

$$H(y) := \frac{s}{r}\mathrm{e}^{-y^2}, \quad y \in (0,y_0],$$

and, in addition, inequality (2.28) is satisfied.

EXAMPLE 2.15. The other example from [16] can also be considered. Let α, τ, p and λ be arbitrary real numbers and let $A_0, B_0 > 0$. Put

$$K(y,y') := A_0(1+y)^\alpha(1+y')^\alpha,$$
$$\gamma(y,y') := B_0K(y',y-y')(1+y)^\tau[(1+y')(1+y-y')]^{-\tau}\mathrm{e}^{\lambda(y^p-(y-y')^p-(y')^p)},$$
$$\beta_{\mathrm{s}}(y,y') := y\nu(y,y')\biggl(\int_{y-y_0}^{y_0} y''\nu(y,y'')\,\mathrm{d}y''\biggr)^{-1},$$

where

$$\nu(y,z) := (1+z)^{\alpha-\tau}(1+y-z)^{\alpha-\tau}\mathrm{e}^{-\lambda(z^p+(y-z)^p)}.$$

Then, with $P \equiv 1$, $Q \equiv 0$ and

$$H(y) := \frac{B_0}{A_0}(1+y)^{-\tau}\mathrm{e}^{-\lambda y^p - y}, \quad y \in (0, y_0],$$

hypotheses (H_2)–(H_6) and inequality (2.28) are satisfied.

3. Stability

We now focus on stability of the equilibria. For this purpose, let us introduce, for any $\varrho > 0$, the spaces

$$X^+ := \{u \in L_1^+;\ V(u) < \infty\} \quad \text{and} \quad X_\varrho^+ := \{w \in X^+;\ M(w) = \varrho\}.$$

If not stated otherwise, X^+ and X_ϱ^+ are equipped with the L_1 topology, turning them into metric spaces. Observe that both X^+ and X_ϱ^+ are positively invariant, and that the map $(t, u^0) \mapsto u(t; u^0)$ defines a semiflow on X^+ and X_ϱ^+ due to theorem 2.1 and proposition 2.7. Moreover, provided (2.28) holds, theorem 2.8 entails that $u_{\alpha(\varrho)}$ is a global attractor for the semiflow generated on X_ϱ^+, where $\alpha(\varrho)$ is chosen such that $M(u_{\alpha(\varrho)}) = \varrho$.

In order to state the next proposition, we define, for $\eta \in \mathbb{R}$,

$$V_\eta(w) := V(w) - |H|_1 - \eta M(w), \quad w \in X^+.$$

PROPOSITION 3.1. *For $\varrho > 0$, choose $\alpha(\varrho) \in \mathbb{R}$ such that $M(u_{\alpha(\varrho)}) = \varrho$. Then $u_{\alpha(\varrho)}$ is the unique minimizer of V on X_ϱ^+ and of $V_{\alpha(\varrho)}$ on X^+. Moreover, for any minimizing sequence (w_j) of V on X_ϱ^+, we have $w_j \to u_{\alpha(\varrho)}$ in X_ϱ^+.*

Proof. For $r > 0$, define

$$f_r(w) := w\biggl(\log\frac{w}{r} - 1\biggr), \quad w \geqslant 0,$$

with $f_r(0) := 0$. Then f_r has at $w = r$ a global minimum for each $r > 0$. For brevity, we put $\alpha := \alpha(\varrho)$. Given $w \in X^+$, we have

$$V_\alpha(w) = \int_0^{y_0} f_{u_\alpha(y)}(w(y))\,\mathrm{d}y \geqslant \int_0^{y_0} f_{u_\alpha(y)}(u_\alpha(y))\,\mathrm{d}y = V_\alpha(u_\alpha),$$

where the inequality is strict if w differs from u_α on a set of non-zero measure. Hence u_α is the unique minimizer of V_α on X^+. Furthermore, since $M(X_\varrho^+) = \{\varrho\}$, it also minimizes V on X_ϱ^+.

Now let (w_j) be a minimizing sequence of V in X_ϱ^+, i.e.

$$\lim V(w_j) = \inf_{w \in X_\varrho^+} V(w) = V(u_\alpha). \tag{3.1}$$

Observing that this implies

$$|f_{u_\alpha(\cdot)}(w_j(\cdot)) - f_{u_\alpha(\cdot)}(u_\alpha(\cdot))|_1 = V_\alpha(w_j) - V_\alpha(u_\alpha) \to 0,$$

we may extract a subsequence (j') such that $f_{u_\alpha(\cdot)}(w_{j'}(\cdot)) \to f_{u_\alpha(\cdot)}(u_\alpha(\cdot))$ a.e. This easily implies $w_{j'} \to u_\alpha$ a.e. From (3.1), lemma 2.2 and the Dunford–Pettis theorem, we deduce that $(w_{j'})$ is relatively weakly compact in L_1. Therefore, there exists a further subsequence (j'') and $w \in L_1$ such that $w_{j''} \to w$ in $L_{1,\mathrm{w}}$. Since V is weakly lower semi-continuous, we get

$$V(w) \leqslant \liminf_{j''} V(w_{j''}) = V(u_\alpha) < \infty,$$

whence $w \in X_\varrho^+$. From the above considerations, we conclude that $w = u_\alpha$. Altogether, we obtain $w_{j''} \to u_\alpha$ in $L_{1,\mathrm{w}}$ and a.e., so that $w_{j''} \to u_\alpha$, from which the assertion follows. □

THEOREM 3.2. *Let $\varrho > 0$ be given and choose $\alpha(\varrho) \in \mathbb{R}$ such that $M(u_{\alpha(\varrho)}) = \varrho$. Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $u^0 \in X_\varrho^+$ with*

$$|u^0 - u_{\alpha(\varrho)}|_1 < \delta \quad \textit{and} \quad V(u^0) < V(u_{\alpha(\varrho)}) + \delta,$$

we have $|u(t;u^0) - u_{\alpha(\varrho)}|_1 < \varepsilon$ for $t \geqslant 0$.

Proof. Due to [2, proposition 4.3], we merely have to show that V is decreasing along orbits (which was done in proposition 2.7) and that $u_{\alpha(\varrho)}$ lies in a 'potential well' with respect to X_ϱ^+, that is, for given small $\varepsilon > 0$, there exists $\sigma(\varepsilon) > 0$ such that $V(w) - V(u_{\alpha(\varrho)}) \geqslant \sigma(\varepsilon)$ for all $w \in X_\varrho^+$ with $|w - u_{\alpha(\varrho)}|_1 = \varepsilon$. But this readily follows from proposition 3.1. □

Define the metric d by

$$d(w,v) := |w - v|_1 + |V(w) - V(v)|, \quad w, v \in X^+.$$

We conclude with a stability result, being a straight consequence of the decrease of V along orbits.

COROLLARY 3.3. *Let $\varrho > 0$ be arbitrary and choose $\alpha(\varrho) \in \mathbb{R}$ such that $M(u_{\alpha(\varrho)}) = \varrho$. Then the equilibrium $u_{\alpha(\varrho)}$ is stable in (X_ϱ^+, d), that is, for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $u^0 \in X_\varrho^+$ with $d(u^0, u_{\alpha(\varrho)}) < \delta$, we have*

$$d(u(t;u^0), u_{\alpha(\varrho)}) < \varepsilon \quad \textit{for } t \geqslant 0.$$

REMARK 3.4. For the case without scattering and shattering, it is shown in the recent paper [17] that

$$V(u(t;u^0)) \to V(u_\alpha) \quad \text{as } t \to \infty,$$

where $M(u_\alpha) = M(u^0)$. Such an improvement of theorem 2.8 would allow us to conclude asymptotic stability of the equilibrium $u_{\alpha(\varrho)}$ in (X_ϱ^+, d).

Acknowledgments

This research consists of an edited extract of the author's PhD thesis submitted to the Universität Zürich. The author gratefully acknowledges the support of his thesis supervisor Professor H. Amann.

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(*Issued 17 August 2004*)